

# ON SOME INEQUALITIES FOR $s$ -LOGARITHMICALLY CONVEX FUNCTIONS IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

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**ABSTRACT.** In this paper, we establish some new Hadamard type inequalities for  $s$ -logarithmically convex functions in the second sense via fractional integrals by using Lemma 1 which has been proved by Sarıkaya et al. in the paper [3].

## 1. INTRODUCTION

The following result is well known in the literature as Hadamard's inequality [1].

**Theorem 1.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . Then*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

The following definitions is well known in the literature:

**Definition 1.** *A function  $f : I \rightarrow \mathbb{R}$ ,  $\emptyset \neq I \subseteq \mathbb{R}$ , where  $I$  is a convex set, is said to be convex on  $I$  if inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

*holds for all  $x, y \in I$  and  $t \in [0, 1]$ .*

In [2], Akdemir and Tunç were introduced the class of  $s$ -logarithmically convex functions in the first and second sense as the following:

**Definition 2.** *A function  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  is said to be  $s$ -logarithmically convex in the first sense if*

$$(1.2) \quad f(\alpha x + \beta y) \leq [f(x)]^{\alpha^s} [f(y)]^{\beta^s}$$

*for some  $s \in (0, 1]$ , where  $x, y \in I$  and  $\alpha^s + \beta^s = 1$ .*

**Definition 3.** *A function  $f : I \subset \mathbb{R}_0 \rightarrow \mathbb{R}_+$  is said to be  $s$ -logarithmically convex in the second sense if*

$$(1.3) \quad f(tx + (1-t)y) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s}$$

*for some  $s \in (0, 1]$ , where  $x, y \in I$  and  $t \in [0, 1]$ .*

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Clearly, when taking  $s = 1$  in Definition 2 or Definition 3, then  $f$  becomes the standard logarithmically convex function on  $I$ .

**Definition 4.** Let  $f \in L_1[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where  $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ . Here is  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

In the case of  $\alpha = 1$ , the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see [3]- [11].

In [3], Sarıkaya *et. al.* proved the following results for fractional integrals.

**Lemma 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \\ &= \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt. \end{aligned}$$

**Theorem 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ &= \frac{b-a}{2(\alpha+1)} \left( 1 - \frac{1}{2^\alpha} \right) [|f'(a)| + |f'(b)|]. \end{aligned}$$

In the present paper, we will establish several Hermite-Hadamard type inequalities for the class of functions whose derivatives in absolute value are  $s$ -logarithmically convex functions in the first and second sense via Riemann-Liouville fractional integral.

## 2. HADAMARD TYPE INEQUALITIES FOR $s$ -LOGARITHMICALLY CONVEX FUNCTIONS

**Theorem 3.** Let  $I \supset [0, \infty)$  be an open interval and  $f : I \rightarrow (0, \infty)$  is differentiable. If  $f' \in L[a, b]$  and  $|f'|$  is  $s$ -logarithmically convex functions in the second sense on

$[a, b]$  for some fixed  $s \in (0, 1]$  and  $\mu, \eta > 0$  with  $\mu + \eta = 1$ , then the following inequality for fractional integrals with  $\alpha > 0$  holds:

$$(2.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ \leq \frac{b-a}{2} \left\{ \int_0^{1/2} \mu [(1-t)^\alpha - t^\alpha]^{\frac{1}{\mu}} dt + \int_{1/2}^1 \mu [t^\alpha - (1-t)^\alpha]^{\frac{1}{\mu}} dt \right. \\ \left. + \eta \times |f'(b)|^{\frac{s}{\eta}} \psi \left( \frac{s}{\eta}, \frac{s}{\eta} \right) \right\}$$

where

$$(2.2) \quad \Psi(\psi) = \begin{cases} 1, & \psi = 1, \\ \frac{\psi-1}{\ln \psi}, & 0 < \psi < 1 \end{cases} \quad \text{and } \psi(u, v) = |f'(a)|^u |f'(b)|^{-v}, \quad u, v > 0.$$

*Proof.* By Lemma 1 and since  $|f'|$  is  $s$ -logarithmically convex functions in the second sense on  $[a, b]$ , we have

$$(2.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| dt \\ \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a)|^{ts} |f'(b)|^{(1-t)s} dt \\ \leq \frac{b-a}{2} \left\{ \int_0^{1/2} [(1-t)^\alpha - t^\alpha] |f'(a)|^{ts} |f'(b)|^{(1-t)s} dt \right. \\ \left. + \int_{1/2}^1 [t^\alpha - (1-t)^\alpha] |f'(a)|^{ts} |f'(b)|^{(1-t)s} dt \right\},$$

for all  $t \in [0, 1]$ . Using the well known inequality  $mn \leq \mu m^{\frac{1}{\mu}} + \eta n^{\frac{1}{\eta}}$ , on the right side of (2.3), we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ \leq \frac{b-a}{2} \left\{ \int_0^{1/2} \mu [(1-t)^\alpha - t^\alpha]^{\frac{1}{\mu}} dt + \int_0^{1/2} \eta |f'(a)|^{\frac{ts}{\eta}} |f'(b)|^{\frac{(1-t)s}{\eta}} dt \right. \\ \left. + \int_{1/2}^1 \mu [t^\alpha - (1-t)^\alpha]^{\frac{1}{\mu}} dt + \int_{1/2}^1 \eta |f'(a)|^{\frac{ts}{\eta}} |f'(b)|^{\frac{(1-t)s}{\eta}} dt \right\} \\ = \frac{b-a}{2} \left\{ \int_0^{1/2} \mu [(1-t)^\alpha - t^\alpha]^{\frac{1}{\mu}} dt + \int_{1/2}^1 \mu [t^\alpha - (1-t)^\alpha]^{\frac{1}{\mu}} dt \right. \\ \left. + \eta \int_0^1 |f'(a)|^{\frac{ts}{\eta}} |f'(b)|^{\frac{(1-t)s}{\eta}} dt \right\}$$

If  $0 < \lambda \leq 1$ ,  $0 < u, v \leq 1$ , then

$$(2.4) \quad \lambda^{uv} \leq \lambda^{uv}.$$

When  $\psi(u, v) \leq 1$ , by (2.4), we get that

$$(2.5) \quad \int_0^1 |f'(a)|^{\frac{s}{\eta}} |f'(b)|^{\frac{(1-t)s}{\eta}} dt \leq \int_0^1 |f'(a)|^{\frac{st}{\eta}} |f'(b)|^{\frac{s(1-t)}{\eta}} dt = |f'(b)|^{\frac{s}{\eta}} \psi\left(\frac{s}{\eta}, \frac{s}{\eta}\right).$$

From (2.3) to (2.5), (2.1) holds.  $\square$

**Remark 1.** If we take  $\alpha = 1$ , in Theorem 3, then the inequality (2.1) become the inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left[ \frac{\mu^2}{\mu+1} + \eta \times |f'(b)|^{\frac{s}{\eta}} \psi\left(\frac{s}{\eta}, \frac{s}{\eta}\right) \right].$$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

**Theorem 4.** Let  $I \supset [0, \infty)$  be an open interval and  $f : I \rightarrow (0, \infty)$  is differentiable. If  $f' \in L[a, b]$  and  $|f'|$  is  $s$ -logarithmically convex functions in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$  and  $\mu, \eta > 0$  with  $\mu + \eta = 1$  and  $p, q > 1$ , then the following inequality for fractional integrals with  $\alpha > 0$  holds:

$$(2.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \leq \frac{b-a}{2(\alpha p + 1)^{\frac{1}{p}}} |f'(b)|^s (\psi(sq, sq))^{\frac{1}{q}}$$

where  $1/p + 1/q = 1$ , and  $\psi(u, v)$  is defined as in (2.2).

*Proof.* By Lemma 1 and since  $|f'|$  is  $s$ -logarithmically convex functions in the second sense on  $[a, b]$ , we have

$$(2.7) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a)|^{ts} |f'(b)|^{(1-t)s} dt$$

for all  $t \in [0, 1]$ . Using the well known Hölder inequality, on the right side of (2.7) and making the change of variable we have

$$(2.8) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \leq \frac{b-a}{2} \left( \int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(a)|^{qt^s} |f'(b)|^{q(1-t)s} dt \right)^{\frac{1}{q}}.$$

It is know that for  $\alpha, t_1, t_2 \in [0, 1]$ ,

$$|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha,$$

therefore

$$(2.9) \quad \int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \leq \int_0^1 |1-2t|^{\alpha p} dt = \frac{1}{\alpha p + 1}.$$

Since  $|f'|$  is  $s$ -logarithmically convex functions on  $[a, b]$  and  $\psi(u, v) \leq 1$ , we obtain

$$(2.10) \quad \int_0^1 |f'(a)|^{qt^s} |f'(b)|^{q(1-t)^s} dt \leq |f'(b)|^{sq} \psi(sq, sq)$$

From (2.8) to (2.10), (2.6) holds.  $\square$

A different approach leads to the following result.

**Theorem 5.** *Let  $I \supset [0, \infty)$  be an open interval and  $f : I \rightarrow (0, \infty)$  is differentiable. If  $f' \in L[a, b]$  and  $|f'|^q$  is  $s$ -logarithmically convex functions in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$  and  $\mu, \eta > 0$  with  $\mu + \eta = 1$  and  $q \geq 1$ , then the following inequality for fractional integrals with  $\alpha > 0$  holds:*

$$(2.11) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ \leq \frac{b-a}{2^{\frac{q-(1-\alpha)(q-1)}{q}}} \left( \frac{2^\alpha - 1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left( \frac{\mu^2}{\alpha + \mu} + \eta |f'(b)|^{\frac{sq}{\eta}} \psi\left(\frac{sq}{\eta}, \frac{sq}{\eta}\right) \right)^{\frac{1}{q}}$$

where  $\psi(u, v)$  is defined as in (2.2).

*Proof.* By Lemma 1 and using the well known power mean inequality, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| dt \\ \leq \frac{b-a}{2} \left( \int_0^1 |(1-t)^\alpha - t^\alpha| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}$$

It is easily check that

$$\int_0^1 |(1-t)^\alpha - t^\alpha| dt = \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right).$$

Since  $|f'|^q$  is  $s$ -logarithmically convex and using the well known inequality  $mn \leq \mu m^{\frac{1}{\mu}} + \eta n^{\frac{1}{\eta}}$ , we obtain

$$\int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)|^q dt \leq \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a)|^{qt^s} |f'(b)|^{q(1-t)^s} dt \\ \leq \int_0^1 |1-2t|^\alpha |f'(a)|^{qt^s} |f'(b)|^{q(1-t)^s} dt \\ \leq \mu \int_0^1 |1-2t|^{\frac{\alpha}{\mu}} dt + \eta \int_0^1 |f'(a)|^{\frac{qt^s}{\eta}} |f'(b)|^{\frac{q(1-t)^s}{\eta}} dt.$$

It is easily check that

$$\mu \int_0^1 |1-2t|^{\frac{\alpha}{\mu}} dt = \mu \frac{1}{\frac{\alpha}{\mu} + 1} = \frac{\mu^2}{\alpha + \mu}.$$

Afterwards, when  $\psi(u, v) \leq 1$ , by (2.4), we get that

$$(2.12) \quad \int_0^1 |f'(a)|^{\frac{qt^s}{\eta}} |f'(b)|^{\frac{q(1-t)^s}{\eta}} dt \leq \int_0^1 |f'(a)|^{\frac{sq}{\eta}} |f'(b)|^{\frac{sq(1-t)}{\eta}} dt = |f'(b)|^{\frac{sq}{\eta}} \psi\left(\frac{sq}{\eta}, \frac{sq}{\eta}\right).$$

Therefore

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{b-}^\alpha f(a) + J_{a+}^\alpha f(b)] \right| \\ & \leq \frac{b-a}{2} \left( \frac{2}{\alpha+1} \left( 1 - \frac{1}{2^\alpha} \right) \right)^{1-\frac{1}{q}} \left( \mu \int_0^1 |1-2t|^{\frac{\alpha}{\mu}} dt + \eta \int_0^1 |f'(a)|^{\frac{qt^s}{\eta}} |f'(b)|^{\frac{q(1-t)^s}{\eta}} dt \right)^{\frac{1}{q}} \\ & \leq \frac{b-a}{2} \left( \frac{2}{\alpha+1} \left( 1 - \frac{1}{2^\alpha} \right) \right)^{1-\frac{1}{q}} \left( \frac{\mu^2}{\alpha+\mu} + \eta |f'(b)|^{\frac{sq}{\eta}} \psi\left(\frac{sq}{\eta}, \frac{sq}{\eta}\right) \right)^{\frac{1}{q}} \end{aligned}$$

which completes the proof.  $\square$

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